SIMILARITY OF NON-NEWTONIAN FLOWS. II.*
AUTOMORPHY, POWER-LAW MODELS AND THE REYNOLDS NUMBER

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The automorphic and power-law rheological models are discussed which enable definition of the Reynolds number having the same physical meaning as the one known from the hydrodynamics of Newtonian liquids. On an example it is demonstrated that such types of rheological models can be constructed, which represent qualitatively also the visco-elastic and thixotropic behaviour of materials.

In our previous work[1] we have made an effort to formulate as generally as possible the theory of similarity of non-Newtonian flows and we limited ourselves to the technically interesting category of isotropic incompressible materials. The primary result, important for a technician facing the problem to model experimentally a certain hydrodynamic situation, are the conditions of exact modelling

$$\mathrm{He} = \varrho U_c^2/\tau_I \equiv \text{idem}\,, \tag{1}$$

$$\mathrm{B} = (U_c/R_c)/D_I \equiv \text{idem}\,, \tag{2}$$

$$\mathrm{St} = U_c t_c/R_c \equiv \text{idem}\,, \tag{3}$$

together with conditions of rheological similarity, *i.e.* $H^+[\ldots] \equiv$ idem, the part of which is $\mathrm{Ve} = D_I \tau_I \equiv$ idem, and conditions of similarity of complementary conditions[1], *i.e.* the conditions of geometrical similarity, kinematic similarity of boundary conditions, symmetry conditions *etc.*

It is obvious that neither the explicit and quantitative formulation of all rheological properties of studied materials is possible, nor fulfilling of all conditions of exact modelling in cases when for an engineering experiment a model liquid must be used. Some aspects of approximate modelling of non-Newtonian flows are therefore discussed here, based on substitutive constitutive relations to which certain functional properties are ascribed.

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GENERALIZED NEWTONIAN LIQUIDS

The category of Generalized Newtonian Fluids[2] (GNF-materials) is the category of idealized non-Newtonian materials with all their rheological properties assumed to be expressed in the form

$$\boldsymbol{\tau} = \eta \mathbf{D}\,, \tag{4}$$

where the dependence of apparent viscosity η,

$$\eta \equiv \tau / D\,, \tag{5}$$

on the second invariant of the tensor of shear rate $D = (\frac{1}{2}\mathbf{D} : \mathbf{D})^{1/2}$ resp. of shear stress $\tau = (\frac{1}{2}\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2}$, expressed in the form

$$\eta = \eta[D] \quad \text{or} \quad \eta = \eta[\tau]\,, \tag{6a, b}$$

is the material property.

The rheological similarity[1] of two GNF-materials is in general stipulated by the possibility to find for them two material constants – the characteristic shear rate D_I, characteristic shear stress τ_I, resp. characteristic viscosity $\eta_I = \tau_I / D_I$ so that dimensionless relations between D/D_I, τ/τ_I resp. η/η_I would be identical. The apparent dimensionless viscosity m, dimensionless shear rate p and dimensionless shear stress ϑ are introduced by relations

$$m = \eta/\eta_I = \vartheta/p\,, \quad \vartheta = \tau/\tau_I\,, \quad p = D/D_I\,, \tag{7a, b, c}$$

according to which the conditions of rheological similarity of GNF-materials can be formulated in one of equivalent forms

$$m[p] \equiv \text{idem}\,, \quad m[\vartheta] \equiv \text{idem}\,, \quad \vartheta[p] \equiv \text{idem}\,. \tag{8a, b, c}$$

As it has been explained in our previous work[1], it is advisable in developing the normalized mathematical flow model, to normalize the quantities included in the differential momentum balance

$$\varrho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \,.\, \nabla \mathbf{v} \right) = - \nabla P + \nabla \,.\, \boldsymbol{\tau} \tag{9}$$

with the use of operating parameters U_c, R_c, P_c, t_c, whenever possible. Especially, the tensor of shear rate and its second invariant are normalized into the form

$$\mathbf{D}^* = \mathbf{D} \,.\, R_c/U_c\,, \quad D^* = DR_c/U_c\,. \tag{10a, b}$$

The rheological constitutive relation (4) can be written in a form enabling its substitution into the relation (9) which is

$$\boldsymbol{\tau} = (\tau_I/D_I)\, m[D/D_I]\, \mathbf{D}\,, \tag{11a}$$

or, with normalized variables, as

$$\tau/\tau_{\mathrm{I}} = \frac{U_{\mathrm{c}}}{R_{\mathrm{c}}D_{\mathrm{I}}}\, m\left[\frac{U_{\mathrm{c}}}{R_{\mathrm{c}}D_{\mathrm{I}}} D^*\right] \mathbf{D}^* . \qquad (11b)$$

The momentum balance for GNF-materials is the result of substitution of Eq. (*11b*) into (*9*) and normalization of the resulting equation by the term τ_{I}

$$\mathrm{He}\left(\frac{1}{\mathrm{St}}\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^* . \nabla^* \mathbf{v}^*\right) = -\,\mathrm{Cr}\,\nabla^* P^* + \mathrm{B}\,\nabla^* . \left(m[\mathrm{B}D^*]\,\mathbf{D}^*\right), \qquad (12)$$

where all quantities with the asterisk "*" are normalized by use of the operating parameters, and where the dimensionless criteria of dynamic similarity are defined by relations (*1*)–(*3*) and (*13*):

$$\mathrm{Cr} = P_{\mathrm{c}}/\tau_{\mathrm{I}} . \qquad (13)$$

Some authors[3,4] introduce the Reynolds number in the form

$$\mathrm{Re}_{\mathrm{I}} = R_{\mathrm{c}} U_{\mathrm{c}} \varrho / \eta_{\mathrm{I}} = \mathrm{He}/\mathrm{B} . \qquad (14)$$

This formulation cannot be considered fully suitable as it gives an impression that the Re_{I} defined in this way has the character of a criterion determining generally the flow regime in the sense of transition of laminar-turbulent regimes, and measuring the effect of inertia forces in comparison with the forces of viscous friction as it is usual in the hydrodynamics of Newtonian liquids. The regime of steady flow for GNF-materials is obviously in general controlled by a pair of dimensionless criteria (He, B) resp. (Re_{I}, B) and by the condition of rheological similarity $m(p) \equiv$ idem.

Both for rheologically similar GNF-materials and for steady flows it is necessary to fulfill two modelling conditions, for inst. (*1*) and (*2*) and thus scalling-up *i.e.* performing the model experiments with the use of original liquid is in general impossible. This is a significant difference from modelling of steady Newtonian flows with the use of Reynolds number in cases where it is not necessary to take into account the dynamics of the free liquid surface (for example for flow in closed channels, for flow through a bed of granular material, in mixing in narrow vessels *etc.*) and where it is sufficient to fulfill the modelling rule $R_{\mathrm{c}} U_{\mathrm{c}} =$ const.

In case of steady flows with negligible effects of inertia forces, when $\mathrm{Re} \ll 1$, the modelling of Newtonian flows is not limited by any dynamic modelling condition while for GNF-materials B = idem must be generally fulfilled *i.e.* scalling-up with the same liquid must be realized at $U_{\mathrm{c}}/R_{\mathrm{c}} =$ const.

Further on we concentrate our attention to such formalisms in description of rheological properties which would permit the mentioned scalling-up and would generally give simpler exact modelling conditions. Rheological models based on such description can, though they do not mostly express the rheological properties of studied materials exactly, be used as the initial mathematical model in formulation of conditions of approximate hydrodynamic similarity[5-8].

Automorphic GNF-Materials

Functional relation $z = f(x)$ between a pair of real positive numbers x, z is automorphic if a function $g(x)$ exists, so that for any arbitrary real x, y

$$f(x \,.\, y) = f(x) \,.\, g(y) \tag{15}$$

holds. By substitution $x = 1$ into relation (*15*) follows that the relation (*15*) can be generally written in a form

$$f(x \,.\, y) = f(x) \,.\, f(y)/f(1) \,. \tag{16}$$

By substitution $y = 1/x$ into relation (*16*) follows

$$f(1/x) = f^2(1)/f(x)\,, \quad \text{and thus as well} \quad f(x/y) = f(1) \,.\, f(x)/f(y)\,. \tag{17a, b}$$

Let the dimensionless viscosity characteristics $m = m_a(p)$ be automorphic. The corresponding dimensional relation can be written according to (*11a*) as

$$\eta[D] = \tau_I/D_I \,.\, m_a[D/D_I]\,, \tag{18a}$$

or, with the use of relation (*17b*), in the form*

$$\eta[D] = \frac{\tau_I m_a[1]}{D_I m_a[D_I]} \,.\, m_a[D]\,, \tag{18b}$$

from which it follows that the automorphic viscosity characteristics has only one material constant K_a

$$K_a = \frac{\tau_I m_a[1]}{D_I m_a[D_I]}\,, \tag{18c}$$

unlike the general case when a non-automorphic viscosity characteristics has two parameters. Trivial example of an automorphic function is given by Newtonian liquids, where m_a = const. and according to the relation (*18c*) $K_a = \tau_I/D_I = \mu$.

* It is known[9,10] that to an arbitrary system of physical quantities the so-called coherent system of units exists which enables to express relations between these quantities in the same form as relations between the numerical values of these quantities, *i.e.* without conversion factors. Therefore it is possible to apply without modifications the definition of automorphic relations introduced for numbers also to relations between physical, dimensional quantities. It is, of course, necessary to ascribe to automorphic relations certain dimensional transformation properties which are obvious at best in special cases presented in the following paragraphs.

By using the properties (*16*), (*17*) of a automorphic viscosity characteristics, the normalized differential momentum balance can be modified into the form

$$\mathrm{Re_a}\left(\frac{1}{\mathrm{St}}\cdot\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^*\,.\,\nabla^*\mathbf{v}^*\right) = -\,\mathrm{Cr_a}\,\nabla^* P^* + \nabla^*.\left(m_a[D^*]\,.\,\mathbf{D}^*\right), \qquad (19)$$

$$\mathrm{Re_a} = \frac{\mathrm{He}\; m_a[1]}{\mathrm{B}\; m_a[\mathrm{B}]} = \frac{\varrho U_c R_c}{K_a m_a[U_c/R_c]}, \qquad (20)$$

$$\mathrm{Cr_a} = \frac{\mathrm{A}\; m_a[1]}{\mathrm{B}\; m_a[\mathrm{B}]} = \frac{P_c}{K_a(U_c/R_c)\; m_a[U_c/R_c]}, \qquad (21)$$

where only $\mathrm{Re_a}$ (resp. only $\mathrm{Cr_a}$) acts as an independent dimensionless criterion.

In hydrodynamic situations when in Newtonian liquids the Reynolds number Re is the only determining dimensionless criterion, for automorphic GNF-materials as well $\mathrm{Re_a}$ is the only one. If the way is compared in which Re and $\mathrm{Re_a}$ are introduced into the normalized differential momentum balance it can be generally expected that $\mathrm{Re_a}$ is similar to Re, *i.e.* that it characterizes the ratio of inertia and internal friction forcess, while in relation (*19*) there is no other independent criterion He to which the same role could be ascribed.

The hydrodynamics of automorphic GNF-materials has, beside the possibility of introducing the generalized Reynolds number another important aspect. In case of creeping flows, $\mathrm{Re_a} \to 0$, the relation (*19*) takes the form

$$\mathrm{Cr_a}\,\nabla^* P^* + \nabla^*\,.\left(m_a[D^*]\,\mathbf{D}^*\right) = 0\,, \qquad (22)$$

from which by solution for given boundary conditions, or by a single experiment the numerical value $\mathrm{Cr_a}$ characterizing the dynamics of creeping flow can be determined. Dependence of P_c on other parameters can in this case be expressed in the form

$$P_c/\mathrm{Cr_a} = K_a(U_c/R_c)\; m_a[U_c/R_c]\,, \qquad (23)$$

i.e. in the same form as the dependence of τ on D in the corresponding automorphic viscosity characteristics

$$\tau = K_a D m_a[D]\,, \qquad (24)$$

while for general GNF-materials the form of dependence

$$\mathrm{Cr} = \mathrm{Cr(B)} \qquad (25)$$

must be determined experimentally resp. by solution of corresponding mathematical model in the whole required range of numbers B resp. of corresponding values U_c/R_c.

POWER-LAW MODEL

The presented results would be undoubtedly very interesting if it were possible to find a sufficiently wide class of automorphic functions expressing the course of viscosity characteristics of real liquids. Let us start with the functional definition *(16)* of the automorphic function and let us try to find sufficiently wide class of their specific courses.

For physical reasons it is obvious that we are limited to continuous and continuously differentiable functions. Differentiation of an automorphic function $f_a(x)$ can be, by successive modifications and by use of its property *(16)*, expressed in the form

$$\frac{df_a(x)}{dx} = \frac{f_a(x)}{x} \cdot \frac{f_a'(1)}{f_a(1)},$$

i.e. as a differential equation

$$\frac{d \ln f_a(x)}{d \ln x} = q = \left. \frac{d \ln f_a(x)}{d \ln x} \right|_{x=1}, \tag{26}$$

whose constant parameter q has the meaning of the value of logarithmic derivative at the point $x = 1$. General solution of this differential equation which represents as well the class of all continuous and continuously differentiable automorphic functions, has the form of a power function with parameters q and k

$$f_a(x) = kx^q. \tag{27}$$

In case of GNF-material we obtain the already known results, *i.e.* the expression of the course of viscosity characteristics in the form

$$m_a(p) = p^{(n-1)}, \tag{28}$$

where $n = q + 1$ is the so-called flow index[5,6], resp. in the form

$$\tau = KD^n, \tag{29}$$

and the formulation of the generalized Reynolds number in the form[5,6,11]

$$Re' = U_c^{2-n} R_c^n \varrho / K \tag{30}$$

and the dimensionless number Cr′ in the form[12]

$$Cr' = P_c / [K(U_c/R_c)^n]. \tag{31}$$

The presented analysis is useful as the resulting relations *(30)*, *(31)* are not obtained only by methods of dimensional analysis but because they result from certain functional properties (automorphy) of the constitutive equation. It further indicates the way how to look for useful forms of constitutive relations also in more complex cases when it is intended to include into the engineering analyses also other more complicated rheological phenomena.

We do not know the way how to find in an arbitrary class of constitutive relations those which would be automorphic in the mentioned sense. But it becomes obvious that by assumptions based only on dimensional analysis at least the formulation of some properties may be obtained the automorphic constitutive relations would or could have.

From the functional point of view it is possible to formulate Re_a by Eq. (*20*) and it is possible for creeping flows to find the dependences between the operating parameters in the form (*23*) with the condition of automorphic properties (*16*) of the viscosity characteristics (*18*). From the dimensional point of view, when we have already found the power-law as the most general automorphic non-linear function, the definitions of Re' by Eq. (*30*) and of the dependently variable criterion Cr' for creeping flows (*31*) are the consequence of the fact that the corresponding constitutive relations include only one dimensional parameter K. It is therefore quite appropriate to ask whether it is possible to formulate constitutive relations with only one dimensional parameter K, which would describe more complicated rheological phenomena than is the simple non-linearity of the viscosity characteristics.

EXAMPLE OF VISCO-ELASTIC AUTOMORPHIC MATERIAL

The theory of viscometric flows[23] postulates the existence of at most three material viscometric functions by the use of which the dynamics of these flows can be fully described. The opinion on physical meaning of these material functions, of the viscosity characteristics,

$$\tau_{12} = \tau[D] \qquad (32a)$$

of the first and second difference of normal stresses

$$\tau_{11} - \tau_{33} = \sigma_1[D]\,, \quad \tau_{22} - \tau_{33} = \sigma_2[D]\,, \qquad (32b, c)$$

is based on the simple shearing flow where index 1 corresponds in the mentioned relations to the Cartesian coordinate x_1 vertical to undeformed mutually sheared planes, analogically x_2 is the Cartesian coordinate in the direction of the velocity of flow, and x_3 is an indifferent coordinate, vertical to x_1 and x_2.

The most general automorphic system of viscosimetric functions, *i.e.* including only a single dimensional parameter has obviously the form

$$\tau[D] = KD^n\,, \quad \sigma_1[D] = \alpha_1 KD^n = \alpha_1\tau[D] \quad \sigma_2[D] = \alpha_2\tau[D]\,, \qquad (33a, b, c)$$

where beside the dimensional constant K the flow index n and other two dimensionless parameters α_1 and α_2 appear. At the first sight the constitutive assumption formulated by relations (*33a,b,c*) seems to be rather forcible because for example the theory of second-order viscoelasticity[14] leads to expression of viscometric functions in the form

$$\tau[D] = \eta_0 D\,, \quad \sigma_1[D] = (\beta + 2\gamma)\, D^2\,, \quad \sigma_2[D] = \beta D^2\,, \qquad (34a, b, c)$$

which obviously are not in relation with equations (*33a*, *b*, *c*) neither for $n = 1$. However, a number of non-Newtonian liquids[15] are known whose viscosity characteristics for very small D agree with relation (*34a*), but for rather large D their course can be expressed by the use of a power-law, $n \neq 1$, in a relatively wide range of shear rates. Therefore, there is no reason why to eliminate the possibility to describe the course of viscometric functions by the system of relations (*33a*, *b*, *c*).

The description of viscoelastic properties of polymer solutions is by a number of engineering oriented authors, *e.g.* Metzner, Uebler and Fong[16] based on the modified Maxwell model

$$\tau + t_0 \frac{\mathrm{D}\tau}{\mathrm{D}t} = \eta \mathbf{D}\,, \qquad (35)$$

where the apparent viscosity η and the relaxation time t_0 are material constants or material functions of the invariant D. The relaxation time is usually found from viscometric experiments, using Truesdell's[17] definition in the form

$$t_0 = (\sigma_1^2[D] + \sigma_2^2[D])^{1/2}/(D\tau[D])\,. \qquad (36)$$

In the case when the automorphy of viscometric material functions according to Eq. (*33*) can be accepted, relation (*36*) takes the form

$$t_0 = (\alpha_1^2 + \alpha_2^2)^{1/2}/D\,. \qquad (37)$$

It is interesting and it supports the possibility to use automorphic relation also for description of more complex rheological phenomena, that experimental dependence of t_0 on D, mentioned by Metzner[16] for three different visco-elastic polymer systems can be approximately correlated according to relation (*37*) as it follows from Fig. 1, where the data published by Metzner[16] are plotted. The automorphic modification of the Maxwell model can be thus written, for example,

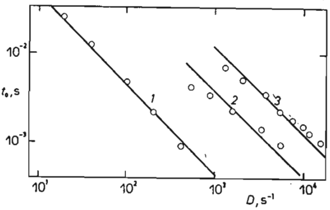

Fig. 1

Dependence of Relaxation Times t_0 on Shear Rate D (according to[16])

1 0·29% Carbopol 941 in water, *2* 0·2% ET 597 in H_2O, *3* 5·0% polyisobutylene in decaline.

in the form

$$\tau + \frac{\alpha}{D} \cdot \frac{D\tau}{Dt} = KD^{n-1}\mathbf{D}, \qquad (38)$$

where α and n are dimensionless parameters and where D, the second invariant of the tensor $\mathbf{D}$, can be replaced by invariants of other kinematic tensors having the dimension of reciprocal time, or their suitable combination.

EXAMPLE OF THIXOTROPHIC AUTOMORPHIC MATERIAL

Fredrickson[18] has recently published a model of thixotrophic behaviour of suspensions. His model, including in the simplest case four constants: η_0 and η_∞ with the dimension of viscosity, λ with the dimension of time and with the dimension of reciprocal stress

$$\tau = \eta\mathbf{D}, \qquad (39)$$

$$\frac{D\eta}{Dt} = F(\eta, D) = -4kD^2\eta^2\left(\frac{\eta - \eta_\infty}{\eta_\infty}\right) + \frac{\eta}{\lambda}\left(\frac{\eta_0 - \eta}{\eta}\right), \qquad (40)$$

foresees a number of phenomena observed with thixotrophic suspensions, namely: *1.* For viscometric flows, when $D\eta/Dt = 0$, the equation $F(\eta, D) = 0$ represents implicitely the course of viscosity characteristic. *2.* Thixotrophic relaxations, *i.e.* hysteresis loops on the dependence of D on τ at programmed time changes of D, resp. τ, and transitions from one viscosity value to another at step changes of shear rate from one non-zero value to another. This transition is usually characterized by exponential functions of time.

When we need the model of thixotrophic behaviour to be automorphic, the function F can be supposed for inst. in the form

$$F[\eta, D] = -\alpha D(\eta - KD^{n-1}). \qquad (41)$$

With the exception that this model fails for $D = 0$, when $F(\eta, D) = 0$, it expresses qualitatively the same phenomena as Fredrickson's model, especially: *1.* For viscometric flows, the viscosity characteristics, *i.e.* the function $F(\eta, D) = 0$ is determined by the power model:

$$\tau = \eta D = KD^n. \qquad (42)$$

2. At a step change of shear rate in the viscometric flow from a non-zero value D_0 to the non-zero value D_1 at the instant $t = 0$, the time course of the instantaneous apparent viscosity on time can be expressed in the form

$$\frac{\eta(t) - KD_0^{n-1}}{K(D_1^{n-1} - D_0^{n-1})} = \exp(-\alpha D_1 t), \qquad (43)$$

which is in agreement with experimentally determined dependences $\eta(t)$ for some of thixotrophic suspensions[18], provided it is possible to express the course of viscosity characteristics in the considered range of shear rate by a power model.

CONCLUSIONS

The conclusions of a theoretical analysis of formal properties of automorphic material functions are: *1.* Automorphy of constitutive relations in an indispensable condition for writing the normalized mathematical flow model is the form including as the only independent criteria of dynamic similarity Re_a and St, eventually other simplexes of geometrical similarity, with only geometrical parameters, and simplexes of rheological similarity with only material constants which can be, however, directly formulated as dimensionless. In case of steady flows of material with automorphic viscosity characteristics Re_a is the only determining criterion of dynamic similarity including operational parameters. In the case when inertial forces can be completely neglected, the formulation of hydrodynamic problem does not include any limitation concerning the values of parameters U_c, R_c, t_c or their ratios. *2.* The most general automorphic relation between two quantities is a power-law. The consequence of automorphy is the possibility to reduce the number of dimensional parameters in the relations between two quantities of different dimensions from two to one.

As the description of non-linear viscosity characteristics by the power model was the only case when in the engineering of non-Newtonian liquids the automorphic or power relation were used, and since the formal properties of automorphic relations seem to us for the above mentioned reasons to be very suitable especially for engineering application, we have presented several other less trivial cases of automorphic formulae. Material derivatives according to time D/Dt in them are generally normalized by invariants of kinematic tensors having the dimension of time *i.e.* for example in the form $(1/D \,.\, D/Dt)$. We have shown that they are able at least qualitatively to describe such rheological phenomena as the existence of normal stresses, visco-elastic and thixotrophic relaxations. Even in these cases the main advantages of automorphic models are preserved which can be considered to be the consequence of existence of the only dimensional parameter K in constitutive relations.

These advantages are for practical purposes especially in the possibility to perform model experiments with the original material on smaller models with the exact modelling conditions preserved because $Re' \equiv$ idem requires only fulfillment of the condition

$$U_c^{2-n} R_c^n = \text{const}$$

and $St \equiv$ idem can be, if necessary, fulfilled by a suitable choice of t_c on the model. In the case when inertial effects can be neglected and the density of liquid is not a parameter which must be included into dimensionless correlations, we can choose U_c as well as R_c without limitations if the inequality $Re' \ll 1$ is fulfilled.

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LIST OF SYMBOLS

D	second invariant of shear rate tensor; arbitrary invariant of kinematic tensors or their combination (s^{-1})
$\mathbf{D}$	shear rate tensor (s^{-1})
D_I	material constant (s^{-1})
D^*	normalized shear rate, Eq. (*10b*) (1)
$\mathbf{D}^*$	normalized shear rate tensor, Eq. (*10a*) (1)
H^+	normalized constitutive functional[1]
K	material constant (dimensional) of the power-law model (dyn cm^{-2} s^{+n})
K_a	constant (dimensional) of automorphic model, Eq. (*18c*) (dyn cm^{-2} s m_a (s))
k	integration constant in Eq. (*27*) (1)
m	dimensionless apparent viscosity Eq. (*7*)
n	flow index, affects dimension of constant K (1)
p	dimensionless shear rate
P	hydrodynamic potential (dyn cm^{-2})
$P^* = P/P_c$	normalized hydrodynamic potential (1)
P_c	dynamic operating parameter[1] (dyn cm^{-2})
R_c	characteristic length of the flow situation (cm)
t	time (s)
t_I	material constant (s)
t_c	operating parameter, characteristic time difference in the basically non-stationary hydrodynamic problems[1], it is dependent on R_c/U_c (s)
t_0	relaxation time defined by Truesdell (s)
$t^* = t/t_c$	normalized time variable (1)
U_c	characteristic velocity (cm s^{-1})
$\mathbf{v}$	velocity (cm s^{-1})
$\alpha, \alpha_1, \alpha_2$	dimensionless material constants in automorphic constitutive relations, simplexes of rheological similarity (1)
β	material constant of constitutive relation of second order viscoelastic material Eq. (*34*) (dyn cm^{-2} s^2)
γ	material constant of constitutive relation of second order viscoelastic material Eq. (*34*) (dyn cm^{-2} s^2)
η_0	material constant of constitutive relation of second order viscoelastic material Eq. (*34*) (dyn cm^{-2} s^2)
η	apparent viscosity, Eq. (*5*), material function (dyn cm^{-2} s)
η_I	material constant (dyn cm^{-2} s)
ϑ	dimensionless shear stress, Eq. (*8c*)
ϱ	density (g cm^{-3})
σ_1, σ_2	first and second difference of normal stresses, material functions (dyn cm^{-2})
τ	second invariant of shear stress tensor, material function (dyn cm^{-2})
τ_I	material constant (dyn cm^{-2})
$\boldsymbol{\tau}$	shear stress tensor, deviator of shear tensor (dyn cm^{-2})
∇	operator Nabla (cm^{-1})
$\nabla^* = R_c \nabla$	normalized operator Nabla (1)

Dimensionless Numbers

B	dynamic number of flow similarity for non-automorphic constitutive relations, Eq. (*2*)

Cr	modified Euler number for creeping flows (Creeping number) for non-automorphic constitutive relations, Eq. (*13*)
Cr_a	Creeping number for automorphic relations, Eq. (*21*)
Cr′	Creeping number for power-law relations, Eq. (*31*)
He	generalized Hedström number, Eq. (*1*)
Re	Reynolds number for non-Newtonian liquids
Re_I	Reynolds number for non-automorphic constitutive relations, Eq. (*14*)
Re_a	Reynolds number for automorphic constitutive relations, Eq. (*20*)
Re′	Reynolds number for power-law relations, Eq. (*30*)
St	Strouhal number for basically non-stationary flows[1], Eq. (*3*)
Ve	viscoelastic number, simplex of rheological similarity for time-dependent liquids[1]

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